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A strong enthalpy formulation for the Stefan problem

A. W. DATE
Mechanical Engineering Department, Indian Institute of Technology, Bombay, India

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Abstract—This paper demonstrates how the condition of constancy with respect to time of the phase-change interface temperature can be incorporated to arrive at a 'strong' enthalpy formulation. The finite-difference solutions obtained with this formulation show that the problem of 'waviness' of the temperature histories encountered with the 'weak' formulation is now removed and accurate solutions are obtained even with a coarse grid irrespective of the time step. The formulation derived requires no 'book-keeping' of the phase-change node, and allows line-by-line integration of the finite-difference equations.

INTRODUCTION

ALL THE previous solutions [1–4] to the Stefan problem have been obtained with what is termed as the 'weak' enthalpy formulation. This formulation can be stated as

$$\rho \frac{\partial H}{\partial t} = K \frac{\partial^2 T}{\partial x^2} \tag{1}$$

with the following conditions at the phase-change interface:

$$T = T_m \quad \text{at} \quad x = x_i \tag{2}$$

and

$$K \frac{\partial T}{\partial x}\bigg|_s - K \frac{\partial T}{\partial x}\bigg|_l = \rho \lambda \frac{\mathrm{d}x_i}{\mathrm{d}t} \tag{3}$$

where x_i is the location of the interface.

Since $\lambda\ \mathrm{d}x_i/\mathrm{d}t$ represents the rate of evolution of the latent heat, it can be readily shown that equation (1) already satisfies the flux condition (3) at the interface.

The implicit finite-difference form of equation (1) for uniform grid spacings is given by

$$H_j^n - H_j^o = \frac{K\Delta t}{\rho \Delta x^2}(T_{j+1}^n + T_{j-1}^n - 2T_j^n) \tag{4}$$

where j identifies the grid node, and superscripts n and o refer to the new and the old values, respectively. The right-hand side of equation (4) can be evaluated from the H–T relations given as follows:

$$H = C_p T_s \qquad \text{for } T < T_m \text{ (solid)} \tag{5}$$

$$H = H_{ps} + C_p T_s \qquad \text{for } T = T_m \text{ (phase change)} \tag{6}$$

$$H = \lambda + C_p T_l \qquad \text{for } T > T_m \text{ (liquid)} \tag{7}$$

where H_{ps} is a pseudo enthalpy defined as

$$\int_t^{t+\Delta t} \frac{\mathrm{d}H_{ps}}{\mathrm{d}t}\,\mathrm{d}t = \lambda = H_l - H_s \tag{8}$$

where H_l and H_s are the liquid and solid enthalpies at the fusion temperature.

Usually, the values of T in equation (4) are replaced by H only for the solid and the liquid nodes. For the phase-change node ($H_s < H < H_l$), however, T is held at T_m. A consequence of this is that, till the interface crosses the control volume surrounding the phase-change node, the nodal temperature is held constant. This yields temperature and heat flux histories which demonstrate a step-like or a wavy pattern.

This pattern can be eliminated in two ways. First, the time step Δt can be iteratively estimated such that the entire control volume liberates the latent heat of the control volume. Voller and Cross [1] term this as the 'node-jumping' scheme; this scheme, however, cannot be extended to the multidimensional problems. The second, and the more obvious alternative is to refine the mesh size (or decrease Δx), so that the time Δt over which the phase-change temperature T is held at T_m becomes very small, and the essentially wavy solution appears smooth. This approach is adopted by Shamsunder [2]. In both the approaches, since values of T are recovered from H–T relations, 'book-keeping' is required to identify the phase-change and the single phase nodes. This precludes the possibility of using the line-by-line integration procedure to solve equation (4).

In the present paper both the above-mentioned problems are eliminated in the following ways.

First the H–T relationship is generalized in such a way that no 'book-keeping' is required; and consequently equation (4) can be solved by the Tridiagonal Matrix Algorithm (TDMA).

Secondly, the problem of waviness is eliminated by satisfying one more condition at the interface, namely

$$\frac{\mathrm{d}T}{\mathrm{d}t}\bigg|_i = 0. \tag{9}$$

NOMENCLATURE

Bi	Biot number, hL/K
C_p	specific heat
h	heat transfer coefficient
H	enthalpy
K	thermal conductivity
L	characteristic length
q^*	dimensionless heat flux, $(St+\theta_{x=0})/St$
St	Stefan number, $C_p(T_m-T_\infty)/\lambda$
T	temperature
t	time
x	x-coordinate.

Greek symbols

α	thermal diffusivity, $\rho C_p/K$
θ	dimensionless temperature
λ	latent heat
ρ	density
Φ	dimensionless enthalpy
Φ'	negative of liquid fraction.

Subscripts

∞	infinity
j	node number
l	liquid or liquidus
m	melting point
s	solid or solidus.

This condition has been used by Goodman [5] to develop closed-form integral solutions to the Stefan problem. The condition has also been used by Lazardis [6] for a two-dimensional Stefan problem, where finite-difference solutions are obtained by what is termed as the 'variable-domain' or 'temperature' based formulation of the Stefan problem.

THE PRESENT CONTRIBUTION

For the purposes of illustration we consider a one-dimensional problem with uniform properties. Equation (1) can then be written in a dimensionless form as

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial^2 \theta}{\partial X^2} \tag{10}$$

where

$$\phi = (H-H_s)/\lambda \tag{11}$$

$$\theta = C_p(T-T_m)/\lambda \tag{12}$$

$$\tau = \alpha t/L^2 \tag{13}$$

$$X = x/L. \tag{14}$$

The solution to equation (10) will be obtained for the problem of solidification of a pure metal initially under saturated liquid state. The solidification is brought about by convective cooling (see Fig. 1). The initial and the boundary conditions are

$$\phi = 1, \quad \theta = 0 \quad \text{at} \quad \tau = 0 \tag{15}$$

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=0} = Bi(St+\theta_{x=0}) \tag{16}$$

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=1} = 0. \tag{17}$$

In terms of the dimensionless variables, the H–T relations can be written as

$$\theta = \phi \quad \text{for } \phi \leqslant 0 \quad \text{(solid)} \tag{18}$$

$$\theta = 0 \quad \text{for } 0 \leqslant \phi \leqslant 1 \quad \text{(phase change)} \tag{19}$$

$$\theta = \phi - 1 \quad \text{for } \phi \geqslant 1 \quad \text{(liquid).} \tag{20}$$

The above three relations are now generalized as

$$\theta = \phi + \phi' \tag{21}$$

where

$$\phi' = 0.5[|1-\phi| - |\phi| - 1]. \tag{22}$$

Note that $\phi' = 0$ in solid, and $\phi' = -1.0$ in liquid. At the phase-change node $\phi' = -\phi$. Now since ϕ represents the liquid fraction of the node, $(1+\phi') = (1-\phi)$ represents the solid fractions.

The implicit finite-difference form of equation (10) can now be written as:

for general node j

$$\phi_j^n[1+2S] = S(\phi_{j-1}^n + \phi_{j-1}^n) + S(\phi'^n_{j+1} - 2\phi_j'^n + \phi'^n_{j-1}) + \phi_j^o \tag{23}$$

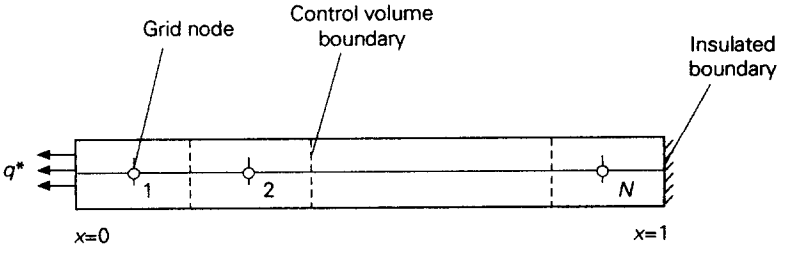

FIG. 1. One-dimensional phase change problem with convective cooling.

for node 1 *near* $X = 0$

$$\phi_1^n[1+S] = S\phi_2^n + S(\phi_2'^n - \phi_1'^n) - Bi(St+\theta_{X=0})S\Delta X + \phi_1^o \quad (24)$$

for last node N *near* $X = 1$

$$\phi_N^n[1+S] = S\phi_{N-1}^n + S(\phi_{N-1}'^n - \phi_N'^n) + \phi_N^o;$$

$$\text{where } S = \Delta\tau/\Delta X^2. \quad (25)$$

It will be readily appreciated that equations (23)–(25) satisfy the Scarborough criterion and are thus unconditionally stable. Further, if ϕ' values are allowed to lag behind ϕ values by one iteration, the equations can be solved by the TDMA. It has already been shown that the TDMA solutions are considerably faster than the point-by-point solutions when fine mesh size is used [7]. The latter, as has already been explained, is a requirement of 'weak' solutions to minimize the waviness problem.

Introduction of generalized relations (21) and (22), however, does not eliminate the waviness problem. To eliminate this problem, an additional condition (9) is now written in dimensionless form as

$$\left.\frac{d\theta}{d\tau}\right|_i = \frac{\partial\theta}{\partial X}\frac{dX_i}{d\tau} + \left.\frac{\partial\theta}{\partial\tau}\right|_i = 0. \quad (26)$$

It can easily be shown that the above condition, along with equation (2) or (19) is satisfied, if the temperature distribution in the solid region $0 \leqslant X \leqslant X_i$ is given by

$$\theta = \theta_{X=0}\left(1 - \frac{X}{X_i}\right). \quad (27)$$

Hence

$$\left.\frac{\partial\theta}{\partial X}\right|_{X=0} = -\theta_{X=0}/X_i. \quad (28)$$

Further, using equation (16), $\theta_{X=0}$ can be recovered as

$$\theta_{X=0} = \frac{-Bi\,St\,X_i}{1+Bi\,X_i} \quad (29)$$

where, since $(1+\phi_j')$ represents the solid fraction, X_i is calculated as

$$X_i = \int_0^1 (1+\phi_j')\,dx. \quad (30)$$

Thus at a given time step, ϕ values (and hence ϕ' values) are taken to be the values corresponding to those calculated at the end of the previous time step. With this initial guess; equations (23)–(25) are solved by TDMA. At the end of iteration, ϕ', X_i and $\theta_{X=0}$ are calculated and the next iteration is performed. Iterations were thus continued till the maximum absolute fractional change in ϕ is less than 10^{-4}.

It should be noted that equation (27) is valid for small Stefan numbers ($St < 0.5$, say); for higher Stefan numbers, a higher order profile may be used as proposed by Goodman [5]. When this is done, evaluation of $\theta_{X=0}$ must be suitably altered.

It is shown in the next section that the above procedure yields non-wavy solutions.

SOLUTIONS

Figure 2 shows the comparison of the solutions obtained by Shamsunder [2] and by Basu and Date [7] with the exact solution for $Bi = 1.0$ and $St = 0.1$. The solutions of Shamsunder show waviness which decreases with decrease in ΔX. Basu and Date also observed waviness which almost vanished at $\Delta X = 0.025$.

Figure 3 shows the results of present computations

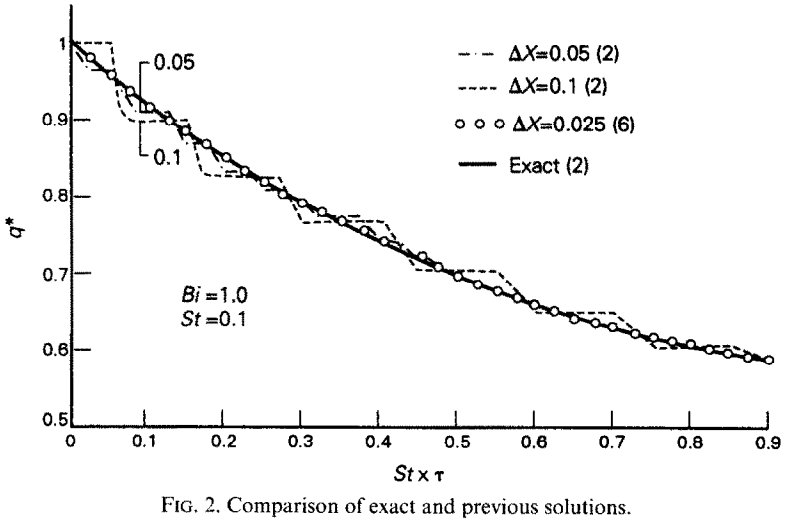

FIG. 2. Comparison of exact and previous solutions.

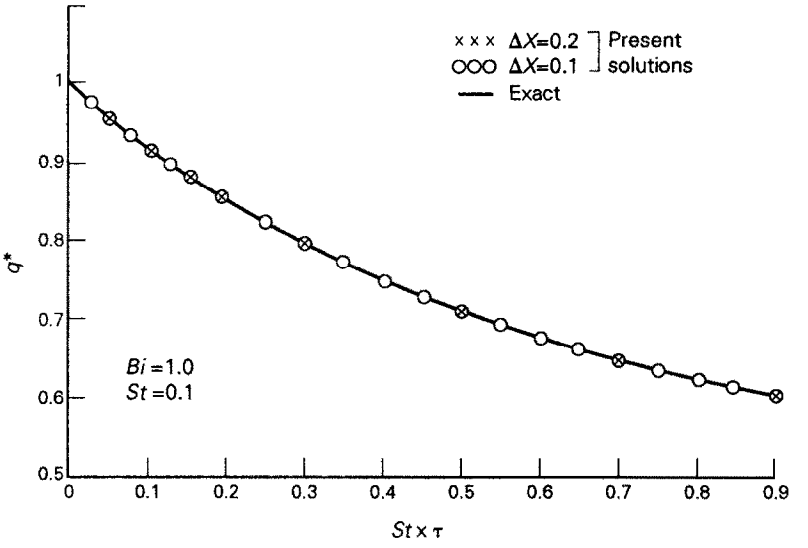

FIG. 3. Comparison of exact and present solutions.

with the strong enthalpy formulation in which $\theta_{x=0}$ is obtained from equation (29). Since this equation recognizes the influence of the location of the interface, extremely accurate and non-wavy solutions are obtained even with $\Delta X = 0.2$. The difference between solutions obtained with $\Delta X = 0.2$ and 0.1 was recognizable only beyond the third decimal place. The solutions were also found to be independent of the time step.

Thus the strong enthalpy formulation presented here has removed the waviness problem besides allowing the use of the Thomas algorithm.

It will be readily recognized that condition (28) is general and can be applied to problems in which either the temperature (i.e. $\theta_{x=0}$) or the flux (i.e. $\partial\theta/\partial x|_{x=0}$) is specified. In the former case, the condition allows evaluation of the gradient at $x = 0$, in the latter it affords the evaluation of the boundary temperature.

CONCLUSIONS

The present paper has thus shown that the waviness problem associated with the 'weak' enthalpy formulation is removed by satisfying the $d\theta/d\tau|_i = 0$ condition at the interface. It is also shown that this condition can be elegantly incorporated through the boundary condition to obtain accurate solutions with considerably coarser grids.

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REFERENCES

1. V. R. Voller and M. Cross, Accurate solutions of moving boundary problems using the enthalpy method, *Int. J. Heat Mass Transfer* **24**, 545–556 (1981).
2. N. Shamsunder, Comparison of numerical methods for diffusion problems with moving boundaries. In *Moving Boundary Problems* (Edited by D. G. Wilson, A. D. Soloman and P. T. Boggs), pp. 165–183. Academic Press, New York (1978).
3. G. E. Bell, The performance of the enthalpy method, *Int. J. Heat Mass Transfer* **25**, 587–589 (1982).
4. A. B. Crowley, Numerical solution of Stefan problems, *Int. J. Heat Mass Transfer* **21**, 215–219 (1978).
5. T. R. Goodman, The heat balance integral and its application to problems involving a change of phase, *Trans. ASME* 355–342 (February 1958).
6. A. Lazardis, A numerical solution of the multidimensional solidification (or melting) problems, *Int. J. Heat Mass Transfer* **13**, 1459–1477 (1970).
7. B. Basu and A. W. Date, On local vs. global implicitness during the solution of melting and solidification problems, *Proc. 9th Heat Mass Transfer Conf.*, Bangalore, India, Paper No. HMT-51-87 (1987).

UNE FORMULATION ENTHALPIQUE FORTE POUR LE PROBLEME DE STEFAN

Résumé—On démontre comment la condition de constance par rapport au temps de la température de l'interface de changement d'état peut être introduite pour arriver à une formulation enthalpique "forte". Les solutions aux différences finies obtenues avec cette formulation montrent que le problème "ondulatoire" des histoires de température rencontrées avec la formulation "faible" est déplacé et que des solutions précises sont obtenues même avec une grille large par rapport au pas de temps. La formulation suivie ne nécessite pas une réservation du noeud de changement de phase et elle fournit une intégration ligne par ligne des équations aux différences finies.

EIN STRENGER ENTHALPIEANSATZ FÜR DAS STEFAN-PROBLEM

Zusammenfassung—Die vorliegende Arbeit zeigt, wie die Bedingung einer zeitlichen Konstanz der Grenzflächentemperatur beim Phasenwechsel so eingebunden werden kann, daß sich ein strenger Enthalpieansatz ergibt. Die Ergebnisse einer Finite-Differenzen-Rechnung aufgrund dieses Ansatzes zeigen, daß das Problem der Welligkeit des zeitlichen Temperaturverlaufs, wie es bei dem weichen Ansatz auftritt, nun beseitigt wird. Selbst bei Verwendung eines weitmaschigen Gitters ergeben sich unabhängig von der Zeitschrittweite genaue Lösungen. Dieser Ansatz benötigt keine "Buchführung" für den Knoten beim Phasenwechsel und erlaubt eine schrittweise Integration der Finite-Differenzen-Gleichungen.

ФОРМУЛИРОВКА ЗАДАЧИ СТЕФАНА НА ОСНОВАНИИ ПОНЯТИЯ "СИЛЬНОЙ" ЭНТАЛЬПИИ

Аннотация—Показано, каким образом можно учитывать условие постоянства во времени температуры межфазной границы при формулировке задачи Стефана с помощью эффективной энтальпии. Решения полученных на основе данной формулировки конечно-разностных уравнений могут быть найдены даже при крупной сетке независимо от временного шага. Выведенная формулировка не требует регистрации узлов фазового перехода при интегрировании конечно-разностных уравнений с переменными коэффициентами.